Statistical resilience of random populations to random perturbations

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We consider populations represented by random collections of real-valued points, and explore their statistical resilience to random perturbations—seeking populations whose statistics remain qualitatively unchanged by the action of arbitrary random perturbations of a certain type. Studying a general physical perturbation scheme, we obtain an explicit characterization of statistically resilient populations, show that these objects are fractal, and comprehensively analyze their topological and statistical structures. An application of statistical resilience attained is an alternative explanation of the ubiquity of power-law statistics.

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I. INTRODUCTION

Populations represented by random collections of real-valued points—the points representing the sizes of the population members—are abundant across the Sciences. Examples include city sizes in a given state, company sizes in a given economy, and node degrees in a given network. Many such populations are impacted, from time to time, by perturbations: a state affected by a demographic change; an economy affected by an industrial or technological change; an ecological network adapting to an environmental change; a social network adapting to a political change. The effect of such perturbations on the impacted populations is stochastic—resulting in a change of the underlying population statistics. In this article we study random perturbations of random populations, and focus on the following question: Are there population statistics which remain qualitatively unchanged by the action of *arbitrary* random perturbations of a certain type?

To formulate this question quantitatively, consider a population $\mathcal{P}$ represented by a countable collection $\{p_k\}_k$ of points taking values in the real range $\mathcal{R}=(r_*,r^*)-r_*$ and r^* denoting, respectively, the ranges's lower and upper bounds $(-\infty \leq r_* < r^* \leq \infty)$. The population is hit by an external random shock which perturbs its points—shifting each population point p_k to a new position $\tilde{p}_k$ in the range $\mathcal{R}$. Thus, the original population $\mathcal{P}=\{p_k\}_k$ is shifted to the perturbed population $\tilde{\mathcal{P}}=\{\tilde{p}_k\}_k$.

Quantitative examples of perturbations $\mathcal{P}\mapsto\tilde{\mathcal{P}}$ include the following: Shift perturbations $\tilde{p}_k=p_k+\xi_k$ on the real line $\mathcal{R}=(-\infty,\infty)$, the shift ξ_k being real valued; multiplicative perturbations $\tilde{p}_k=p_k\xi_k$ on either the positive half-line $\mathcal{R}=(0,\infty)$, or the negative half-line $\mathcal{R}=(-\infty,0)$, the factor ξ_k being positive valued; power-law perturbations $\tilde{p}_k=p_k^{\xi_k}$ on the either the ray $\mathcal{R}=(1,\infty)$, or the unit interval $\mathcal{R}=(0,1)$, the exponent ξ_k being positive valued.

In the aforementioned quantitative examples—considering the perturbation parameters $\{\xi_k\}_k$ to be independent and identically distributed (IID) copies of a random parameter ξ—our goal is to characterize the cases in which the statistics of the perturbed population $\tilde{\mathcal{P}}$ are qualitatively equal to the statistics of the original population $\mathcal{P}$—the qualitative equality holding for *arbitrary* random perturbation parameters ξ.

We coin populations whose statistics are qualitatively invariant to the action of arbitrary random perturbations of a certain type "statistically resilient." A particular case of statistical resilience with respect to multiplicative perturbations on the positive half-line was recently studied in [1], in the context of *Paretian Poisson processes*. And, a particular case of statistical resilience with respect to power-law perturbations on the unit interval was recently studied in [2], in the context of the intrinsic fractality of classic *shot noise* systems.

This article establishes a general theory of statistical resilience of random populations to random perturbations. We consider a general physical perturbation scheme—which, in particular, accommodates the aforementioned quantitative examples as special cases—and study the statistical resilience of random populations with respect to this scheme. The statistically resilient populations obtained turn out to be *infinite* objects with *fractal* features, and a comprehensive quantitative analysis of their topological and statistical structures is presented.

An application of the general theory is an alternative explanation of the ubiquity of *power-law statistics* [3]: In systems where growth is multiplicative and the environment is random and ever changing, the only possible "universal statistics"—which remain statistically resilient to the change of times—are power laws. This explanation bears similarities with the explanation of Zipf's law for city sizes [4] presented in [5], and may apply to diverse issues including the aforementioned city sizes, financial market fluctuations [6], and biological networks [7,8].

The article is organized as follows. Section II describes the setting—the statistics of the population $\mathcal{P}$, and the structure of the general perturbation scheme. The notions of *statistical resilience* and *statistical self-similarity*—with regard to the general perturbation scheme considered—are defined and explored, respectively, in Secs. III and IV. The *renormal-*

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ization and the *topology* of statistically resilient populations are studied, respectively, in Secs. V and VI. Section VII addresses the special case of *power-law statistics*, and Sec. VIII discusses the infiniteness of statistically resilient populations.

II. SETTING

A. Population statistics

The random population $\mathcal{P}$ is considered a *Poisson process* on the range $\mathcal{R}$ with *intensity* $\lambda(x)$ $(x \in \mathcal{R})$ [9]. Informally, this means that the infinitesimal interval $(x, x+dx)$ contains a single point with probability $\lambda(x)dx$, and is empty with probability $1-\lambda(x)dx$ (independently of all other infinitesimal intervals). More precisely, this means that (a) the number of points residing in an interval $I \subset \mathcal{R}$ is Poisson distributed with mean $\int_I \lambda(x)dx$; (b) the number of points residing in disjoint intervals are independent random variables.

Two Poisson processes on the range $\mathcal{R}$ are said to be *statistically equal* if they share the same intensity. We define two Poisson processes on the range $\mathcal{R}$ as *statistically similar* if their intensities are either equal, or differ by a multiplicative factor. Statistical similarity defined here—in the context of Poisson processes—can be regarded as a stochastic version of geometric similarity.

Poisson processes constitute the statistical model for the random scattering of points in general domains [9], and have a wide spectrum of applications ranging from insurance and finance [10] to queueing systems [11]. In recent years we applied Poisson processes in statistical physics to (a) study anomalous statistics displayed by nonlinear shot noise systems [12], and by linear shot noise systems with random relaxations [13]; (b) explore fractality in the context of random populations [14,15], and in the context of probability laws defined on the positive half-line [16]; (c) analyze the correlation cascades of random processes driven by Lévy noises [17], and of nonlinear shot noise systems [18]; (d) obtain nonlinear stochastic limit laws for random populations [19].

In all the aforementioned research articles, the modeling of random populations by Poisson processes turned out to yield results which lay beyond the realm of the more conventional "IID modeling"—i.e., the modeling of random populations by IID random variables. As shall be demonstrated below, this will also be the case in this research.

B. Perturbation scheme

The perturbation is considered to be caused by an external force field applied to all points of the population $\mathcal{P}$, for a certain period of time. The force field propagates the points of the population $\mathcal{P}$, within the range $\mathcal{R}$, according to the ordinary differential equation (ODE) dynamics

$$\dot{\Gamma}(t) = vF(\Gamma(t)) \tag{1}$$

$(t \geq 0)$. The function $F(x)$ $(x \in \mathcal{R})$ represents the force field. The parameter v $(v \neq 0)$ represents the velocity of a population point propagated by the force field.

The function $F(x)$ is assumed positive valued. The primitive of its reciprocal—a function $G(x)$ satisfying $G'(x) = 1/F(x)$ $(x \in \mathcal{R})$—is further assumed to be a monotone-increasing function mapping the range $\mathcal{R}$ onto the real line $(-\infty, \infty)$. These assumptions imply that the solution of the ODE (1)—with initial condition $\Gamma(0)=p$ $(p \in \mathcal{R})$—is given by the trajectory $\Gamma(t)=G^{-1}(G(p)+vt)$ $(t \geq 0)$. Note that if the velocity v is positive then the trajectory $\Gamma(t)$ increases monotonically to the range's upper bound $[\lim_{t\to\infty}\Gamma(t)=r^*]$, and if the velocity v is negative then the trajectory $\Gamma(t)$ decreases monotonically to the range's lower bound $[\lim_{t\to\infty}\Gamma(t)=r_*]$. Henceforth, we refer to the functions $F(x)$ and $G(x)$, respectively, as the *force function* and as the *generator* of the perturbation.

While the action of the force field is considered deterministic, the *response* of the population points to the field's action is considered *stochastic*. Specifically, the velocity of the kth population point is considered a real-valued random variable V_k. Hence, if the force field is applied for t time units then the kth population point is propagated to

$$p_k \mapsto p_k(t) = G^{-1}(G(p_k) + V_k t), \tag{2}$$

and the entire population $\mathcal{P}$ is propagated to $\mathcal{P} \mapsto \mathcal{P}_t = \{p_k(t)\}_k$. The points' random velocities $\{V_k\}_k$ are assumed to be IID copies of a real-valued "generic" random velocity V.

The "displacement theorem" of the theory of Poisson processes ([9], Sec. 5.5; see also the Appendix) implies that the propagated population $\mathcal{P}_t$ is a Poisson process on the range $\mathcal{R}$. Henceforth, we denote by $\lambda_t(x)$ $(x \in \mathcal{R})$ the intensity of the propagated population $\mathcal{P}_t$.

III. RESILIENCE

Consider a perturbation caused by applying the external force field for t time units. The kth population point p_k is thus perturbed to $\tilde{p}_k = p_k(t)$, the population $\tilde{\mathcal{P}}$ is perturbed to $\tilde{\mathcal{P}} = \mathcal{P}_t$, and the perturbed population $\tilde{\mathcal{P}}$ is a Poisson process on the range $\mathcal{R}$ with intensity $\tilde{\lambda}(x) = \lambda_t(x)$ $(x \in \mathcal{R})$. The quantitative perturbation examples given in the introduction are special cases of the general perturbation scheme of Eq. (2)—see Table I.

We define the population $\tilde{\mathcal{P}}$ as *statistically invariant* to the perturbation's action if the perturbed population $\tilde{\mathcal{P}}$ is *statistically equal* to the original population $\mathcal{P}$—the statistical equality holding for *all* perturbation velocities V. Analysis shows (see the Appendix, Sec. 1) that the population $\mathcal{P}$ is statistically invariant to the perturbation's action if and only if its intensity is of the form

$$\lambda_{\text{inv}}(x) = \frac{c}{F(x)}, \tag{3}$$

where c is a positive-valued coefficient. The invariant intensities of Eq. (3) are the *fixed points*—with respect to *all* perturbation velocities V—of the Poissonian intensity mapping $\lambda(x) \mapsto \tilde{\lambda}(x)$. Note that the invariant intensities of Eq. (3) are inversely proportional to the force function $F(x)$—implying that a statistically invariant population $\mathcal{P}$ is dense where the perturbing force is weak, and is sparse where the perturbing force is strong.

TABLE I. Representation of the quantitative perturbation examples given in the introduction as special cases of the general perturbation scheme of Eq. (2). The details of each perturbation example $p_k \mapsto \tilde{p}_k$ are specified: the range $\mathcal{R}$ of the perturbation; the force function $F(x)$ governing the ODE dynamics of Eq. (1); the solution trajectory $\Gamma(t)$, with initial condition $\Gamma(0)=p$, of the ODE dynamics of Eq. (1); the generator $G(x)$ of the perturbation; the connection between the perturbation parameters $\{\xi_k\}_k$ and the perturbation velocities $\{V_k\}_k$.

Perturbation	$\tilde{p}_k=$	Range $\mathcal{R}=$	Force $F(x)=$	Trajectory $\Gamma(t)=$	Generator $G(x)=$	$\xi_k=$
1. Shift	$p_k+\xi_k$	$(-\infty,\infty)$	1	$p+vt$	x	$V_k t$
2. Multiplicative	$p_k\xi_k$	$(0,\infty)$	x	$p\exp(vt)$	$\ln(x)$	$\exp(V_k t)$
3. Multiplicative	$p_k\xi_k$	$(-\infty,0)$	$-x$	$p\exp(-vt)$	$-\ln(-x)$	$\exp(-V_k t)$
4. Power law	$p_k^{\xi_k}$	$(1,\infty)$	$x\ln(x)$	$p^{\exp(vt)}$	$\ln[\ln(x)]$	$\exp(V_k t)$
5. Power law	$p_k^{\xi_k}$	$(0,1)$	$-x\ln(x)$	$p^{\exp(-vt)}$	$-\ln[-\ln(x)]$	$\exp(-V_k t)$

We define the population $\mathcal{P}$ as *statistically resilient* to the perturbation's action if the perturbed population $\tilde{\mathcal{P}}$ is *statistically similar* to the original population $\mathcal{P}$—the statistical similarity holding for *all* perturbation velocities V. Analysis shows (see the Appendix, Sec. 1) that the population $\mathcal{P}$ is statistically resilient to the perturbation's action if and only if its intensity is of the form

$$\lambda_{\text{res}}(x) = \frac{c}{F(x)} \exp[\varepsilon G(x)], \tag{4}$$

where c is a positive-valued coefficient, and where ε is a real-valued exponent. Moreover, if $\lambda(x)=\lambda_{\text{res}}(x)$ then $\tilde{\lambda}(x) = \langle \exp(-\varepsilon t V)\rangle \lambda_{\text{res}}(x)$. Namely, the multiplicative factor by which the intensity of the perturbed population $\tilde{\mathcal{P}}$ differs from the intensity of the original population $\tilde{\mathcal{P}}$ is given by $\langle \exp(-\varepsilon t V)\rangle$—the moment generating function of the perturbation velocity V, evaluated at the point $-\varepsilon t$. The resilient intensities of Eq. (4) are the *eigenintensities*—with respect to *all* perturbation velocities V—of the Poissonian intensity mapping $\lambda(x) \mapsto \tilde{\lambda}(x)$ (which is a linear mapping).

Comparing the characterizations of statistical invariance and statistical resilience obtained, we conclude that the former is a special case of the latter, corresponding to the zero-exponent case $\varepsilon=0$. The intensities of statistically invariant and statistically resilient populations—in case of the quantitative perturbation examples given in the introduction—are presented in Table II.

The motivation behind the aforementioned definitions of statistical invariance and statistical resilience is the following. The *type* of the perturbation—represented by the external force field, and governed by the perturbation generator $G(x)$—depends on the underlying *physics* of the system considered (accommodating the population $\mathcal{P}$), and is fixed and deterministic. The *response* of the population members to the perturbation—quantified by the perturbation velocity V—varies from perturbation to perturbation, and is random. Thus, in order to characterize statistical invariance and statistical resilience—to the action of a certain type of perturbation—we need do so with respect to *all* possible perturbation velocities V.

IV. SELF-SIMILARITY

Consider the general perturbation scheme of Eq. (2), and observe the population $\mathcal{P}$ as it is propagated in time—yielding the population trajectory $\{\mathcal{P}_t; t \geq 0\}$. We define the population $\mathcal{P}$ as *statistically self-similar* with respect to the perturbation's action if *all* propagated populations $\{\mathcal{P}_t\}_{t>0}$ are *statistically similar* to each other. Analysis shows (see the Appendix, Sec. 1) that the population $\mathcal{P}$ is statistically self-similar with respect to the perturbation's action if and only if its intensity admits the resilient form $\lambda(x)=\lambda_{\text{res}}(x)$ of Eq. (4)—in which case the intensity of the t-propagated population $\mathcal{P}_t$ is given by $\lambda_t(x)=\langle \exp(-\varepsilon t V)\rangle \lambda_{\text{res}}(x)$.

Combining this result together with the result of the preceding section we conclude that the population $\mathcal{P}$ is statistically resilient to the perturbation's action if and only if it is

TABLE II. The intensities $\lambda_{\text{inv}}(x)$ and renormalization functions $R_k(x)$ of statistically invariant populations, and the intensities $\lambda_{\text{res}}(x)$ and renormalization functions $R_k(x)$ of statistically resilient populations—in the cases of the quantitative perturbation examples given in Table I.

Perturbation	Invariance $\lambda_{\text{inv}}(x)=$	Renormalization $R_k(x)=$	Resilience $\lambda_{\text{res}}(x)=$	Renormalization $R_k(x)=$
1. Shift	c	kx	$c\exp(\varepsilon x)$	$x+\ln(k)/\varepsilon$
2. Multiplicative	cx^{-1}	x^k	$cx^{\varepsilon-1}$	$xk^{1/\varepsilon}$
3. Multiplicative	$c(-x)^{-1}$	$-(-x)^k$	$c(-x)^{-\varepsilon-1}$	$xk^{-1/\varepsilon}$
4. Power law	$c[\ln(x)]^{-1}x^{-1}$	$\exp[\ln(x)]^k$	$c[\ln(x)]^{\varepsilon-1}x^{-1}$	$x^{k^{1/\varepsilon}}$
5. Power law	$c[-\ln(x)]^{-1}x^{-1}$	$\exp-[-\ln(x)]^k$	$c[-\ln(x)]^{-\varepsilon-1}x^{-1}$	$x^{k^{-1/\varepsilon}}$

statistically self-similar with respect to the perturbation's action. Hence, the notions of statistical resilience and statistical self-similarity coincide. The notion of statistical resilience, however, is conceptually static—whereas the notion of statistical self-similarity is applicable to both static and dynamic settings.

We considered a static setting in which at time 0 we were given a static input—the population $\mathcal{P}$—that was thereafter propagated by the action of the external force field. Consider now a counterpart dynamic setting in which the population points are introduced dynamically in time, rather than being all statically present at time 0. Specifically, consider the following *shot noise* system model—whose setting is analogous to the static setting of Sec. II, and which is based on the nonlinear shot noise system model presented in [12].

Shots of random magnitudes arrive to the system—via an external Poissonian inflow—stochastically in time. Shot magnitudes take values in the real range $\mathcal{R}$, and shots of magnitude x arrive with Poissonian inflow intensity $\lambda(x)$ ($x \in \mathcal{R}$). A shot arriving to the system is propagated by the ODE dynamics of Eq. (1), with random velocity V. Thus, a shot of magnitude x, τ time units after having arrived to the system, is propagated to the random position $G^{-1}(G(x)+V\tau)$. The shots are propagated independently of each other, i.e., the shots' velocities are IID random variables.

The shot noise system is initiated at time 0. At time t ($t>0$) the shots present in the system—originating from shots arriving to the system during the time interval $[0,t]$, and propagated by the ODE dynamics of Eq. (1)—form a population of shots $\mathcal{S}_t$ scattered randomly across the range $\mathcal{R}$. The "displacement theorem" of the theory of Poisson processes ([9], Sec. 5.5; see also the Appendix) implies that the shot population $\mathcal{S}_t$ is a Poisson process on the range $\mathcal{R}$, and we denote its intensity by $\eta_t(x)$ ($x \in \mathcal{R}$).

We define the shot noise system as *statistically self-similar* if all shot populations $\{\mathcal{S}_t\}_{t>0}$ are *statistically similar* to each other. Analysis shows (see the Appendix, Sec. 2) that the shot noise system is statistically self-similar if and only if its inflow intensity admits the resilient form $\lambda(x)=\lambda_{\text{res}}(x)$ of Eq. (4)—in which case (a) the intensity of the shot population $\mathcal{S}_t$ is given by $\eta_t(x)=a(t)\lambda_{\text{res}}(x)$; (b) the self-similarity factor $a(t)$ is given by

$$a(t)=\begin{cases} t & \text{if } \varepsilon=0, \\ \left\langle \dfrac{1-\exp(-t\varepsilon V)}{\varepsilon V} \right\rangle & \text{if } \varepsilon\neq 0 \end{cases} \tag{5}$$

($t>0$). Hence, we conclude the following: Be the setting static or dynamic—statistical self-similarity is characterized by the statistically resilient intensities $\lambda_{\text{res}}(x)$ of Eq. (4).

V. RENORMALIZATION

We turn now to study the *renormalization* of statistically resilient populations, and do so using the Poissonian renormalization scheme introduced in [14] (devised there in order to explore fractality in the context of Poisson processes).

Consider a random population $\mathcal{P}$ on the range $\mathcal{R}$. The k superposition of the population $\mathcal{P}$ is the union population $\mathcal{P}_1\cup\cdots\cup\mathcal{P}_k$, where the populations $\mathcal{P}_1,\cdots,\mathcal{P}_k$ are k IID copies of $\mathcal{P}$. We seek a k-order *renormalization function* $R_k(x)$ ($x\in\mathcal{R}$)—a monotone increasing function which maps the range $\mathcal{R}$ onto itself—for which the renormalized population

$$\mathcal{P}^{(k)}=\{R_k(p)\}_{p\in\mathcal{P}_1\cup\cdots\cup\mathcal{P}_k} \tag{6}$$

is *statistically equal* to the population $\mathcal{P}$. Namely, we seek a k-order shift $p\mapsto R_k(p)$ of the k-superposition population points ($p\in\mathcal{P}_1\cup\cdots\cup\mathcal{P}_k$) that will *statistically retrieve* the initial population $\mathcal{P}$.

Analysis shows (see the Appendix, Sec. 3) that if the population $\mathcal{P}$ is statistically resilient to the perturbation's action—its intensity admitting the resilient form $\lambda_{\text{res}}(x)$ of Eq. (4)—then the function

$$R_k(x)=\begin{cases} G^{-1}(kG(x)) & \text{if } \varepsilon=0, \\ G^{-1}\left(G(x)+\dfrac{\ln(k)}{\varepsilon}\right) & \text{if } \varepsilon\neq 0 \end{cases} \tag{7}$$

is a k-order renormalization function. Note that the renormalization functions of Eq. (7) satisfy the composition rule $R_{k_1}(R_{k_2}(x))=R_{k_1k_2}(x)$ ($k_1,k_2=1,2,\ldots$). This composition rule implies that the renormalization obtained is *self-consistent*: A k_1-order renormalization followed by a k_2-order renormalization is identical to a k_1k_2-order renormalization.

The renormalization functions of statistically resilient populations—in the cases of the quantitative perturbation examples given in the introduction—are presented in Table II.

VI. TOPOLOGY

The intensities $\lambda_{\text{res}}(x)$ of statistically resilient populations are nonintegrable over the range $\mathcal{R}$. Namely, $\int_{r_*}^{r^*}\lambda_{\text{res}}(x)dx=\infty$. This nonintegrability implies that statistically resilient populations are *infinite* and non-IID objects: A statistically resilient Population consists of infinitely many points which do *not* form a collection of IID random variables. The "non-IID" structure of statistically resilient populations renders the results obtained in this research beyond the realm of IID modeling.

The *topological structure* of statistically resilient populations turns out to be determined by the *sign* of their exponent ε, and can be either double sided, increasing, or decreasing.

Double-sided topology. In this case (a) The exponent ε is zero; (b) the intensity $\lambda_{\text{res}}(x)$ is integrable neither at the lower bound r_*, nor at the upper bound r^*; (c) the population points can be ordered monotonically via a double-sided sequence of order statistics $\cdots<O_{-2}<O_{-1}<O_{+1}<O_{+2}<\cdots$ which diverge to the range's lower and upper bounds: $\lim_{n\to-\infty}O_n=r_*$ and $\lim_{n\to\infty}O_n=r^*$.

Increasing topology. In this case (a) the exponent ε is positive; (b) the intensity $\lambda_{\text{res}}(x)$ is integrable at the lower bound r_*, and nonintegrable at the upper bound r^*; (c) the population points can be ordered monotonically via an increasing sequence of order statistics $O_1<O_2<O_3<\cdots$ which diverge to the range's upper bound: $\lim_{n\to\infty}O_n=r^*$.

Decreasing topology. In this case (a) The exponent ε is negative; (b) the intensity $\lambda_{\mathrm{res}}(x)$ is nonintegrable at the lower bound r_*, and integrable at the upper bound r^*; (c) the population points can be ordered monotonically via a decreasing sequence of order statistics $\cdots < O_3 < O_2 < O_1$ which diverge to the range's lower bound: $\lim_{n\to\infty} O_n = r_*$.

Parametrizing statistically resilient populations—with a common perturbation generator $G(x)$—by their exponent ε, we obtain that a topological phase transition takes place at the exponent value $\varepsilon=0$. At this critical value the population points accumulate at both the lower bound r_* and upper bound r^*. In the exponent range $\varepsilon<0$ accumulation holds only at the lower bound r_*, whereas in the exponent range $\varepsilon>0$ accumulation holds only at the upper bound r^*.

In the reminder of this section we address additional issues regarding the topological structure of statistically resilient populations: Extreme points, simulation of the order statistics, and the inverse problem.

A. Extreme points

Statistically resilient populations with nonzero exponents ($\varepsilon \neq 0$) posses an *extreme point*—the order statistic O_1. This order statistic is the population's *minimal point* in the exponent range $\varepsilon > 0$, and is the population's *maximal point* in the exponent range $\varepsilon < 0$. Analysis shows (see the Appendix, Sec. 4) that the probability law of the extreme point O_1 is given by

$$\exp\left(-\frac{c}{|\varepsilon|}\exp[\varepsilon G(x)]\right) = \begin{cases} P(O_1 > x) & \text{if } \varepsilon > 0, \\ P(O_1 \le x) & \text{if } \varepsilon < 0 \end{cases} \tag{8}$$

($x \in \mathcal{R}$).

Both the statistically resilient intensity $\lambda_{\mathrm{res}}(x)$ of Eq. (4), and the extreme probability law of Eq. (8) are uniquely determined by the perturbation generator $G(x)$—this fact implying a one-to-one correspondence between these two functions. The one-to-one correspondence, in turn, establishes an *extremal characterization* of statistically resilient populations. In case of the quantitative perturbation examples given in the introduction the extremal characterization is as follows: (1) Shift perturbations on the real line—*Gumbel* maxima; (2) multiplicative perturbations on the positive half-line—*Fréchet* maxima; (3) multiplicative perturbations on the negative half-line—*Weibull* maxima; (4) power-law perturbations on the ray $(1,\infty)$—*Hyper-Pareto* minima; (5) power-law perturbations on the unit interval—*Hyper-beta* maxima.[1]

B. Simulation of the order statistics

The *simulation* of the sequence of order statistics of statistically resilient populations is given by the following Monte Carlo algorithm:

$$\begin{cases} O_{\pm n} = G^{-1}\left(\dfrac{\pm 1}{c}[\mathcal{E}_{\pm 1} + \cdots + \mathcal{E}_{\pm n}]\right) & \text{if } \varepsilon = 0, \\ O_n = G^{-1}\left[\dfrac{1}{\varepsilon}\ln\left(\dfrac{|\varepsilon|}{c}[\mathcal{E}_1 + \cdots + \mathcal{E}_n]\right)\right] & \text{if } \varepsilon \neq 0 \end{cases} \tag{9}$$

($n=1,2,\ldots$), where $\{\cdots,\mathcal{E}_{-2},\mathcal{E}_{-1},\mathcal{E}_{+1},\mathcal{E}_{+2},\cdots\}$ and $\{\mathcal{E}_1,\mathcal{E}_2,\cdots\}$ are sequences of independent and exponentially distributed random variables with unit mean.

The simulation algorithm of Eq. (9) is easily implementable and highly efficient. Moreover, from this simulation algorithm it is straightforward to obtain, in closed form, the probability law of each order statistic. The proof of the simulation algorithm is analogous to the proof of Proposition 1 in [21].

C. The inverse problem

So far we considered the general perturbation scheme of Eq. (2) as given, and deduced the properties of the corresponding statistically resilient populations. In this section we consider the inverse problem—in which a statistically resilient population is observed, and the underlying perturbation scheme is to be deduced. Specifically, we are given the intensity $\lambda_{\mathrm{res}}(x)$ of a statistically resilient population, and wish to infer the perturbation's force field—i.e., the force function $F(x)$ governing the ODE dynamics of Eq. (1).

Analysis shows (see the Appendix, Sec. 4) that the force function $F(x)$ is given—up to a multiplicative factor—by the following topology-contingent reconstruction formula:

$$F(x) = \begin{cases} \dfrac{1}{\lambda_{\mathrm{res}}(x)} & \text{if } \varepsilon = 0, \\ \dfrac{1}{\lambda_{\mathrm{res}}(x)}\displaystyle\int_{r_*}^{x}\lambda_{\mathrm{res}}(x')dx' & \text{if } \varepsilon > 0, \\ \dfrac{1}{\lambda_{\mathrm{res}}(x)}\displaystyle\int_{x}^{r^*}\lambda_{\mathrm{res}}(x')dx' & \text{if } \varepsilon < 0 \end{cases} \tag{10}$$

[the ε classification regards the topology of the statistically resilient population observed—double-sided ($\varepsilon=0$), increasing ($\varepsilon>0$), or decreasing ($\varepsilon<0$)]. The reconstruction formula of Eq. (10) is also a *reverse-engineering* formula: It specifies the underlying dynamics required—represented by the force function $F(x)$—in order to yield a statistically resilient population with predesired intensity $\lambda_{\mathrm{res}}(x)$.

VII. POWER-LAW STATISTICS

Power-law statistics—manifesting a power-law connection between measurements and their occurrence frequencies—are abundant and ubiquitously observed across the Sciences. Examples range from web sites and book sales to moon-crater diameters and earthquake magnitudes [3].

In the context of integer-valued measurements, power-law statistics are often referred to as "Zipf's law"—named after the linguist George Kingsley Zipf who observed such statistics in word frequencies [22]. If X is an integer-valued measurement, then Zipf's law is characterized by power-law statistics of the form $P(X=n)=[1/\zeta(\nu+1)]\, n^{-\nu-1}$ $(n=1,2,\ldots)$,

[1]Gumbel, Fréchet, and Weibull are the extreme-value probability laws of *extreme value theory*—the only possible stochastic limit laws of affine-scaled maxima of sequences of IID random variables [20]. Hyper-Pareto and Hyper-beta are nonlinear *fractal probability laws* [16].

where ν is a positive-valued exponent, and where $\zeta(\cdot)$ denotes the ζ function.

In the context of positive-valued measurements, power-law statistics are often referred to as "Pareto's law"—named after the economist Vilfredo Pareto who observed such statistics in income distributions [23]. If X a positive-valued measurement, then Pareto's law is characterized by power-law statistics of the form $P(X \in dx) = (\nu l^{\nu}) x^{-\nu-1} dx \ (x > l)$, where ν is a positive-valued exponent, and where l is a positive-valued lower bound.

Note that in both Zipf's and Pareto's laws the exponent ν is restricted to the positive range—in order to ensure normalization (i.e., probabilities summing up to unity). Moreover, in Pareto's law normalization further requires the presence of a lower bound—a "cutoff" prohibiting the measurement from ranging over the entire positive half-line $(0, \infty)$.

A population of positive-valued measurements can be modeled either as a sequence of IID random variables, or as a Poisson process. In the former case power-law statistics are represented by Pareto's law; in the latter case they are represented by power-law intensities of the form $\lambda(x) = c x^{\varepsilon - 1}$ $(x > 0)$, where ε is a real-valued exponent, and where c is a positive-valued constant. The Poissonian modeling—in sharp contrast to the IID modeling—does not require normalization: The intensity $\lambda(x)$ need not be integrable over the positive half-line. Hence, in the Poissonian modeling (a) the exponent ε is real valued—in contrast to the positive-valued Paretian exponent ν; (b) the measurements range over the entire positive half-line—no lower bound cutoff l is required.

We already encountered power-law intensities of the form $\lambda(x) = c x^{\varepsilon - 1}$ $(x > 0)$. Indeed, in the case of multiplicative perturbations on the positive half-line we obtained that $\lambda_{\mathrm{res}}(x) = c x^{\varepsilon - 1}$ $(x > 0)$ (Table II, second example). On the other hand, setting $\lambda_{\mathrm{res}}(x) = c x^{\varepsilon - 1}$ $(x > 0)$ into the reconstruction formula of Eq. (10) yields the force function $F(x) = x$ $(x > 0)$—corresponding to multiplicative perturbations on the positive half-line. Thus, in the context of Poisson processes on the positive half-line, we conclude that *power-law statistics characterize statistical resilience with respect to multiplicative perturbations*.

This conclusion is an alternative explanation of the prevalence and ubiquity of power-law statistics across the Sciences: *In systems where growth is multiplicative and the environment is random and ever changing, the only possible "universal statistics"—which remain statistically resilient to the change of times—are power laws*. The quintessential example of such systems are financial markets—which indeed exhibit power-law statistics [6]. An explanation similar in spirit—in the context of Zipf's law for city sizes [4]—is presented in [5].

VIII. DISCUSSION

As noted already, statistically resilient populations are infinite objects consisting of infinitely many points. In reality however, populations—cities in a given state, companies in a given economy, nodes in a given network, etc.—are always finite objects. We turn now to discuss this discrepancy.

Fractal objects—be they deterministic or stochastic—are, by definition, infinite objects. Indeed, a fractal object is characterized by some kind of self-similarity ranging over all scales [24]. This implies that in order to facilitate the notion of self-similarity, an infinite range of scales is implicitly required.

As a deterministic example, consider the Koch snowflake—the snowflake-shaped domain whose boundary is the Koch curve [25]. The Koch snowflake is the aggregate of infinitely many triangles scaling down in size. This infinite aggregate is necessary a structure for the geometric self-similarity of the Koch snowflake.

As a stochastic example, consider Brownian motion—the jagged random trajectory of diffusive motion [26]. Brownian motion is a random superposition of infinitely many functions scaling down in size [27]. This infinite superposition is necessary a structure for the statistical self-similarity of Brownian motion.

Yet, in reality, neither snowflakes nor diffusion trajectories are infinite objects. In reality, Physics always imposes a lower-bound resolution level, or "cutoff." The Koch curve and Brownian motion are mathematical idealizations—based on the notion of "self-similarity over all scales"—of finite real-world objects. Statistically resilient populations should be understood in the very same way: Mathematical idealizations—based on the notion of statistical resilience—of finite real-world populations.

This is well exemplified by the statistical power laws observed in city sizes [4] and in financial market fluctuations [6]. Would the underlying populations be infinite, then power-law statistics would be observed on all scales. Yet, power-law statistics are observed only at the "upper tail" of the empirical distribution—i.e., above a cutoff induced by the finiteness of real-world populations.

So why bother with infinite populations in the first place? To understand fractals the notion of self-similarity was introduced—a mathematical idealization which implicitly assumes infinitely many scales. The very same thing takes place in this research: To understand statistical structures which withstand random perturbations we introduced the notion of statistical resilience—a mathematical idealization which implicitly assumes infinite populations. In both idealizations infiniteness is required in order to facilitate our mathematical definition and understanding. Having gained the mathematical insight, we can "get back to reality" using a physical cutoff.

Equation (10) reconstructs the force function $F(x)$ of the perturbation scheme underlying a given statistically resilient population, based on the population's statistically resilient intensity $\lambda_{\mathrm{res}}(x)$. Consider now the reconstruction of the force function $F(x)$ based on an empirical observation—consisting of finitely many points—of the statistically resilient population. The empirical reconstruction of the force function $F(x)$ is given by the function $1/S'(x)$ where $S'(x)$ is the derivative of the function $S(x)$ which, in turn, is constructed from the empirical data as follows:

Double-sided topology. In this case (a) the empirical observations are restricted to the subrange (l, u), where l is a lower-bound cutoff and u is an upper-bound cutoff $(r_* < l < u < r^*)$; (b) set $N(x)$ to be the number of population points observed below the level x $(l < x < u)$; (c) set $S(x)$ to be smoothening of the function $N(x)/N(u)$ $(l<x<u)$.

Increasing topology. In this case (a) the empirical observations are restricted to the subrange (r_*,u), where u is an upper-bound cutoff $(r_*<u<r^*)$; (b) set $N(x)$ to be number of population points observed below the level x $(r_*<x<u)$; (c) set $S(x)$ to be smoothening of the function $\ln[N(x)/N(u)]$ $(r_*<x<u)$.

Decreasing topology. In this case (a) the empirical observations are restricted to the subrange (l,r^*), where l is a lower-bound cutoff $(r_*<l<r^*)$; (b) set $N(x)$ to be the number of population points observed above the level x $(l<x<r^*)$; (c) set $S(x)$ to be a smoothening of the function $-\ln[N(x)/N(l)]$ $(l<x<r^*)$.

IX. CONCLUSION

We considered general populations represented by real-valued Poisson processes, perturbed by the general physical perturbation scheme of Eq. (2). Within this setting, the notions of *statistical resilience* and *statistical self-similarity* were introduced, and the classes of *statistically resilient* and *statistically self-similar* populations were characterized. These population classes were shown to coincide, and their topological and statistical structures were comprehensively analyzed. We concluded with a specific application of the general theory: An alternative explanation of the ubiquity of power-law statistics.

APPENDIX

Henceforth, the acronym PDF stands for "probability density function." In the proofs we shall use the "displacement theorem" of the theory of Poisson processes ([9], Sec. 5.5):

Let Π be a Poisson process on a Euclidean domain $\mathcal{X}$ with intensity $\lambda(x)$ $(x\in\mathcal{X})$, and let $\mathcal{Y}$ be another Euclidean domain. Transform each point of the process Π—independently of all other points—as follows: If located at x, transform it to the random point $Y_x\in\mathcal{Y}$, where $P(Y_x\in dy)=\psi(x;y)dy$.[2] Set $\tilde{\Pi}$ to be the set of transformed points. Then, $\tilde{\Pi}$ is a Poisson process on a Euclidean domain $\mathcal{Y}$ with intensity

$$\tilde{\lambda}(y)=\int_{\mathcal{X}}\lambda(x)\psi(x;y)dx \quad (y\in\mathcal{Y}). \tag{A1}$$

1. Resilience and self-similarity

Consider the stochastic map

$$x\mapsto Y_x=G^{-1}(G(x)+Vt) \tag{A2}$$

where (i) t is a positive parameter (time), and V is a real-valued random variable (the generic velocity) with PDF $\psi_V(v)$ (v real); (ii) the deterministic input point x and the random output point Y_x are in the range $\mathcal{R}$. A straightforward calculation implies that the PDF of the random output point Y_x is given by

$$\psi_t(x;y)=\psi_V\left(\frac{G(y)-G(x)}{t}\right)\frac{1}{tF(y)} \quad (y\in\mathcal{R}). \tag{A3}$$

The aforementioned displacement theorem of the theory of Poisson processes implies that if the population $\mathcal{P}$ is a Poisson process with intensity $\lambda(x)$ $(x\in\mathcal{R})$ then the population $\mathcal{P}_t$ is a Poisson process with intensity

$$\lambda_t(y)=\int_{\mathcal{R}}\lambda(x)\psi_t(x;y)dx \quad (y\in\mathcal{R}). \tag{A4}$$

Substituting the PDF of Eq. (A3) into Eq. (A4), using the change of variables $v=[G(y)-G(x)]/t$, and setting $H(\theta)=\lambda(G^{-1}(\theta))F(G^{-1}(\theta))$ and $H_t(\theta)=\lambda_t(G^{-1}(\theta))F(G^{-1}(\theta))$ (θ real), we obtain that

$$H_t(G(y))=\int_{-\infty}^{\infty}H(G(y)-vt)\psi_V(v)dv \quad (y\in\mathcal{R}). \tag{A5}$$

Statistical resilience holds if and only if $\lambda_t(x)=m\lambda(x)$ $(x\in\mathcal{R})$, where m is a multiplicative factor dependent on the time parameter t and on the random velocity V. In turn, $\lambda_t(x)=m\lambda(x)$ $(x\in\mathcal{R})$ holds if and only if $H_t(\theta)=mH(\theta)$ (θ real). Equation (A5), however, implies that $H_t(\theta)=mH(\theta)$ can hold if and only if the function H is an exponential $H(\theta)=c\exp(\varepsilon\theta)$ (θ real; c positive, ε real)—in which case $\lambda(x)=\lambda_{\mathrm{res}}(x)$ $(x\in\mathcal{R})$ and $m=\langle\exp(-\varepsilon tV)\rangle$. Statistical invariance is a special case of statistical resilience, corresponding to $\varepsilon=0$.

Statistical self-similarity holds if and only if $\lambda_t(x)=a(t)\lambda_1(x)$ $(x\in\mathcal{R},\ t>0)$, where $a(t)$ is a multiplicative factor dependent on the time parameter t and on the random velocity V. In turn, $\lambda_t(x)=a(t)\lambda_1(x)$ $(x\in\mathcal{R},\ t>0)$ holds if and only if $H_t(\theta)=a(t)H_1(\theta)$ (θ real, $t>0$). Equation (A5), however, implies that $H_t(\theta)=a(t)H_1(\theta)$ can hold if and only if the function H is an exponential $H(\theta)=c\exp(\varepsilon\theta)$ (θ real; c positive, ε real)—in which case $\lambda_t(x)=\lambda_{\mathrm{res}}(x)$ $(x\in\mathcal{R})$ and $a(t)=\langle\exp(-\varepsilon tV)\rangle$.

2. Shot noise

Consider the stochastic map

$$(s,x)\mapsto Y_{(s,x)}=G^{-1}(G(x)+V(t-s)) \tag{A6}$$

where, (i) t is a positive parameter (time), and V is a real-valued random variable (the generic velocity) with PDF $\psi_V(v)$ (v real); (ii) $0\leq s\leq t$, $x\in\mathcal{R}$, and $Y_{(s,x)}\in\mathcal{R}$. The stochastic map of Eq. (A6) represents the random magnitude $Y_{(s,x)}$, at time t, of a shot with initial magnitude x, arriving to the system at time s. A straightforward calculation implies that the PDF of the random magnitude $Y_{(s,x)}$ is given by

$$\psi_t(s,x;y)=\psi_V\left(\frac{G(y)-G(x)}{t-s}\right)\frac{1}{(t-s)F(y)} \quad (y\in\mathcal{R}). \tag{A7}$$

The aforementioned displacement theorem of the theory of Poisson processes implies that if shots of magnitude x

[2] For each $x\in\mathcal{X}$ the function $\psi(x;\cdot)$ is a PDF on $\mathcal{Y}$.

arrive with Poissonian inflow intensity $\lambda(x)$ ($x \in \mathcal{R}$) then the shot population $\mathcal{S}_t$ is a Poisson process with intensity

$$\eta_t(y) = \int_0^t \int_{\mathcal{R}} \lambda(x)\psi_t(s,x;y)dxds \quad (y \in \mathcal{R}, t > 0). \tag{A8}$$

Substituting the PDF of Eq. (A7) into Eq. (A8), using the change of variables $v=[G(y)-G(x)]/(t-s)$ and $u=(t-s)$, and setting $H(\theta)=\lambda(G^{-1}(\theta))F(G^{-1}(\theta))$ and $H_t(\theta) = \eta_t(G^{-1}(\theta))F(G^{-1}(\theta))$ (θ real), we obtain that

$$H_t(G(y)) = \int_0^t \int_{-\infty}^{\infty} H(G(y) - vu)\psi_V(v)dvdu \quad (y \in \mathcal{R}, t > 0). \tag{A9}$$

Statistical self-similarity holds if and only if $\eta_t(x) = a(t)\eta_1(x)$ ($x \in \mathcal{R}$, $t>0$), where $a(t)$ is a multiplicative factor dependent on the time parameter t and on the random velocity V. In turn, $\eta_t(x)=a(t)\eta_1(x)$ ($x \in \mathcal{R}$, $t>0$) holds if and only if $H_t(\theta)=a(t)H_1(\theta)$ (θ real, $t>0$). Equation (A9), however, implies that $H_t(\theta)=a(t)H_1(\theta)$ can hold if and only if the function H is an exponential $H(\theta)=c\exp(\varepsilon\theta)$ (θ real; c positive, ε real)—in which case $\lambda_t(x)=\lambda_{\mathrm{res}}(x)$ ($x \in \mathcal{R}$) and

$$a(t) = \int_0^t \langle \exp(-\varepsilon uV)\rangle du \quad (t > 0). \tag{A10}$$

Equation (A10), in turn, yields Eq. (5).

3. Renormalization

Combining together the "superposition theorem" ([9], Sec. II B) and the aforementioned displacement theorem of the theory of poisson processes, implies that if the population $\mathcal{P}$ is a Poisson process with intensity $\lambda(x)$ ($x \in \mathcal{R}$) then the population $\mathcal{P}^{(k)}$ is a poisson process with intensity

$$\lambda^{(k)}(x) = k\frac{\lambda(R_k^{-1}(x))}{R_k'(R_k^{-1}(x))} \quad (x \in \mathcal{R}). \tag{A11}$$

Let $\Lambda(x)$ ($x \in \mathcal{R}$) be a primitive of the intensity $\lambda(x)$ [i.e., $\Lambda'(x)=\lambda(x)$], and set

$$R_k(x) = \Lambda^{-1}(k\Lambda(x)) \quad (x \in \mathcal{R}). \tag{A12}$$

It is straightforward to check that substituting the function of Eq. (A12) into Eq. (A11) yields $\lambda^{(k)}(x)=\lambda(x)$—implying that the population $\mathcal{P}^{(k)}$ is statistically equal to the population $\mathcal{P}$. Hence, the function of Eq. (A12) is a *renormalization function*.

A primitive of the statistically resilient intensity $\lambda_{\mathrm{res}}(x)$ ($x \in \mathcal{R}$) of Eq. (4) is

$$\Lambda_{\mathrm{res}}(x) = \begin{cases} cG(x) & \text{if } \varepsilon = 0, \\ \dfrac{c}{\varepsilon}\exp(\varepsilon G(x)) & \text{if } \varepsilon \neq 0 \end{cases} \tag{A13}$$

($x \in \mathcal{R}$). Substituting Eq. (A13) into Eq. (A12) yields the renormalization function of Eq. (7).

4. Extremes and inversion

Double-sided topology. Consider the case $\varepsilon=0$. Equation (10) (case $\varepsilon=0$) follows trivially from Eq. (3).

Increasing topology. Consider the case $\varepsilon>0$, and note that

$$\int_{r_*}^x \lambda_{\mathrm{res}}(x')dx' = \frac{c}{\varepsilon}\exp[\varepsilon G(x)] \quad (x \in \mathcal{R}). \tag{A14}$$

In this case the population's minimal point O_1 is greater than the level x ($x \in \mathcal{R}$) if and only if the population has no points in the subrange $(r_*, x]$—an event occurring with probability

$$\exp\left(-\int_{r_*}^x \lambda_{\mathrm{res}}(x')dx'\right). \tag{A15}$$

Combining together Eqs. (A14) and (A15) yields Eq. (8) (case $\varepsilon>0$). Equation (A14) also implies that

$$G(x) = \frac{1}{\varepsilon}\ln\left(\frac{\varepsilon}{c}\int_{r_*}^x \lambda_{\mathrm{res}}(x')dx'\right) \quad (x \in \mathcal{R}). \tag{A16}$$

Differentiating equation (A16) further implies that

$$\frac{1}{F(x)} = c\frac{\lambda_{\mathrm{res}}(x)}{\int_{r_*}^x \lambda_{\mathrm{res}}(x')dx'} \quad (x \in \mathcal{R}), \tag{A17}$$

which, in turn, yields Eq. (10) (case $\varepsilon>0$).

Decreasing topology. Consider the case $\varepsilon<0$, and note that

$$\int_x^{r^*} \lambda_{\mathrm{res}}(x')dx' = -\frac{c}{\varepsilon}\exp[\varepsilon G(x)] \quad (x \in \mathcal{R}). \tag{A18}$$

In this case the population's maximal point O_1 is no greater than the level x ($x \in \mathcal{R}$) if and only if the population has no points in the subrange (x, r^*)—an event occurring with probability

$$\exp\left(-\int_x^{r^*} \lambda_{\mathrm{res}}(x')dx'\right). \tag{A19}$$

Combining together Eqs. (A18) and (A19) yields Eq. (8) (case $\varepsilon<0$). Equation (A19) also implies that

$$G(x) = \frac{1}{\varepsilon}\ln\left(-\frac{\varepsilon}{c}\int_x^{r^*} \lambda_{\mathrm{res}}(x')dx'\right) \quad (x \in \mathcal{R}). \tag{A20}$$

Differentiating Eq. (A20) further implies that

$$\frac{1}{F(x)} = c\frac{\lambda_{\mathrm{res}}(x)}{\int_x^{r^*} \lambda_{\mathrm{res}}(x')dx'} \quad (x \in \mathcal{R}), \tag{A21}$$

which, in turn, yields Eq. (10) (case $\varepsilon<0$).

[1] I. Eliazar and J. Klafter, J. Stat. Phys. **131**, 487 (2008).
[2] I. Eliazar, Phys. Rev. E **77**, 061103 (2008).
[3] M. E. J. Newman, Contemp. Phys. **46**, 323 (2005).
[4] F. Auerbach, Petermanns Geographische Mitteilungen **59**, 74 (1913).
[5] X. Gabaix, Q. J. Econ. **114**, 739 (1999).
[6] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, Nature (London) **423**, 267 (2003).
[7] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A. L. Barabasi, Nature (London) **407**, 651 (2000).
[8] H. Jeong, S. Mason, A. L. Barabasi, and Z. N. Oltvai, Nature (London) **411**, 41 (2001).
[9] J. F. C. Kingman, *Poisson Processes* (Oxford University Press, Oxford, 1993).
[10] P. Embrechts, C. Kluppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance* (Springer, New York, 1997).
[11] R. W. Wolff, *Stochastic Modeling and the Theory of Queues* (Prentice-Hall, London, 1989).
[12] I. Eliazar and J. Klafter, Proc. Natl. Acad. Sci. U.S.A. **102**, 13779 (2005).
[13] I. Eliazar, Phys. Rev. E **76**, 041128 (2007).
[14] I. Eliazar and J. Klafter, Physica A **383**, 171 (2007).
[15] I. Eliazar, Physica A **386**, 318 (2007).
[16] I. Eliazar and J. Klafter, Phys. Rev. E **77**, 061125 (2008).
[17] I. Eliazar and J. Klafter, J. Phys. A **40**, F307 (2007).
[18] I. Eliazar and J. Klafter, Phys. Rev. E **75**, 031108 (2007).
[19] I. Eliazar and J. Klafter, Physica A **387**, 4985 (2008).
[20] J. Galambos, *Asymptotic Theory of Extreme Order Statistics*, 2nd ed. (Krieger, New York, 1987).
[21] I. Eliazar, Prob. Eng. Inform. Sci. **19**, 289 (2005).
[22] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, 1949).
[23] V. Pareto, *Cours D'Économie Politique* (Droz, Geneva, 1896).
[24] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[25] H. von Koch, *Arkiv for Matematik* 1, 681 (1904).
[26] A. Einstein, Ann. Phys. **17**, 132 (1905).
[27] N. Wiener, J. Math. Phys. (Cambridge, Mass.) **2**, 131 (1923).